Projections of random fractals and measures and Liouville quantum gravity

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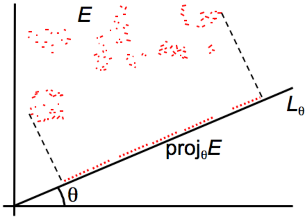
Projections of sets

We will work in $\mathbb{R}^2$ throughout this talk.



Let proj_θ denote orthogonal projection from $\mathbb{R}^2$ to the line L_θ, let $\dim_H$ be Hausdorff dimension, let $\mathcal{L}$ be Lebsegue measure on L_θ.

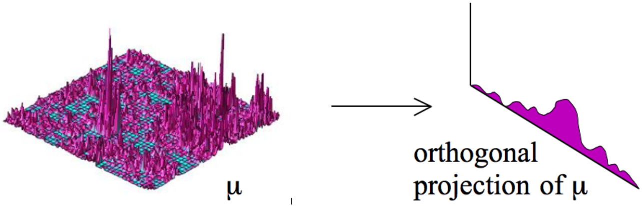
Theorem (Marstrand 1954) Let $E \subset \mathbb{R}^2$ be a Borel set with $\dim_H E > 1$. Then for Lebesgue almost all $\theta \in [0, \pi)$,

$$\mathcal{L}(\mathrm{proj}_\theta E) > 0.$$

Projections of measures

Write $\dim_H \mu = \inf\{\dim_H E : \mu(E) > 0\}$ for the (lower) Hausdorff dimension of measure μ.
We project measures in the obvious way:

$$(\text{proj}_\theta \mu)(A) = \mu\{x : \text{proj}_\theta \in A\} \text{ for } A \subset L_\theta.$$



Theorem (Marstrand/Kaufman) Let μ be a Borel measure on $\mathbb{R}^2$. If $\dim_H \mu > 1$ then $\text{proj}_\theta \mu$ is absolutely continuous w.r.t Lebesgue measure for almost all θ, in fact with L^2 density, i.e. there is $f \in L^2$ such that $\text{proj}_\theta \mu(A) = \int_A f(x)dx$ for $A \subset L_\theta$.

Exceptional directions

These theorems tell us nothing about which particular directions have projections with $\mathcal{L}(\mathrm{proj}_\theta E) = 0$ or $\mathrm{proj}_\theta \mu$ not absolutely continuous.

However, the set of exceptional directions can't be 'too big':

Theorem (F, 1982) If $E \subseteq \mathbb{R}^2$ and $\dim_H E > 1$,

$$\dim_H\{\theta : \mathcal{L}(\mathrm{proj}_\theta E) = 0\} \leq 2 - \dim_H E.$$

General problem: Find classes of sets where all projections have positive length, and measures where all projections are absolutely continuous (or better), or at least where there are few exceptional directions.

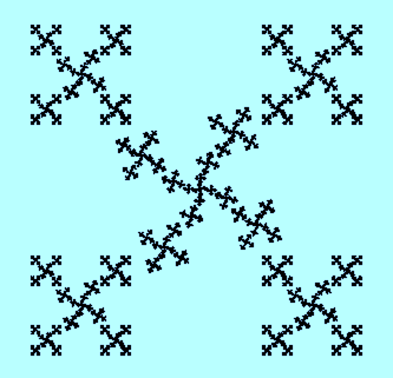
Self-similar sets

Given an iterated function system of contracting similarities $f_1, \ldots, f_m : \mathbb{R}^2 \to \mathbb{R}^2$ there exists a unique non-empty compact $E \subset \mathbb{R}^2$ such that

$$E = \bigcup_{i=1}^{m} f_i(E)$$

which we call a self-similar set.

The family $\{f_1, \ldots, f_m\}$ has dense rotations if the rotational component of at least one of the f_i is an irrational multiple of π.



A self-similar set with dense rotations

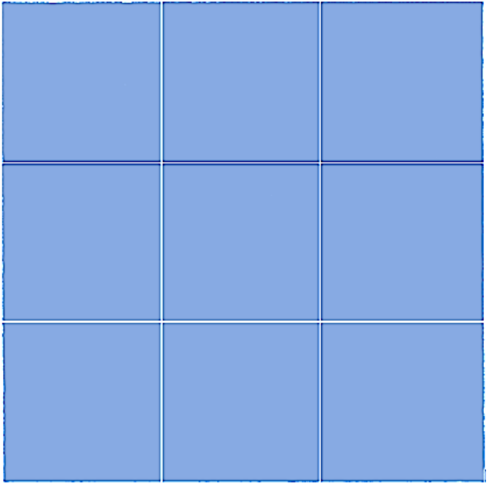
Projections of positive length

Theorem (Shmerkin & Solomyak 2014) Let $E \subset \mathbb{R}^2$ be the self-similar attractor of an IFS with dense rotations with $\dim_H E > 1$. Then $\mathcal{L}(\mathrm{proj}_\theta E) > 0$ for all θ except (perhaps) for a set of θ of Hausdorff dimension 0.

This is a corollary of an analogous result for the absolute continuity of projections of self-similar measures.

The proof uses the 'Erdös-Kahane' method.

Mandelbrot percolation on a square



- Squares are repeatedly divided into $M \times M$ subsquares
- Each square is retained independently with probability p ($\simeq 0.6$).

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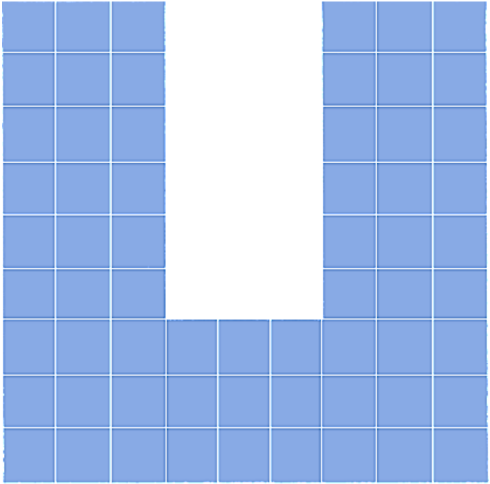

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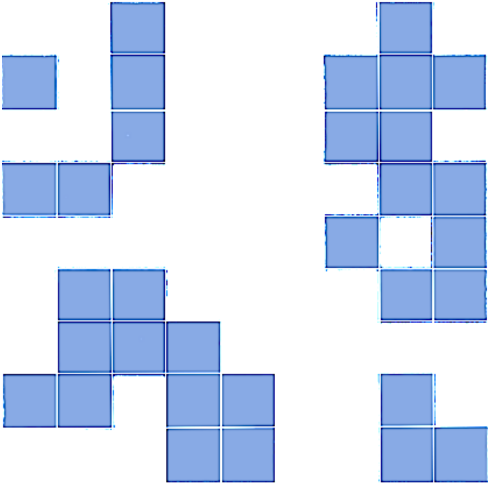

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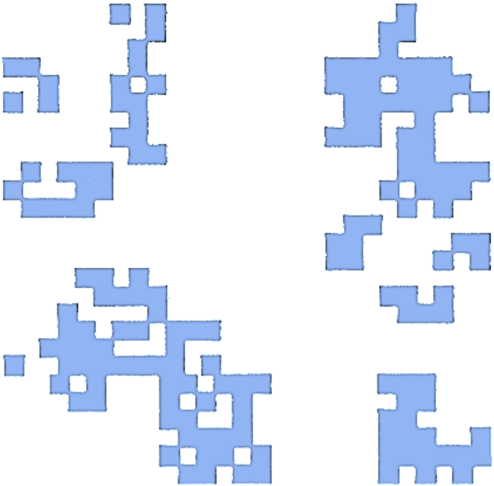

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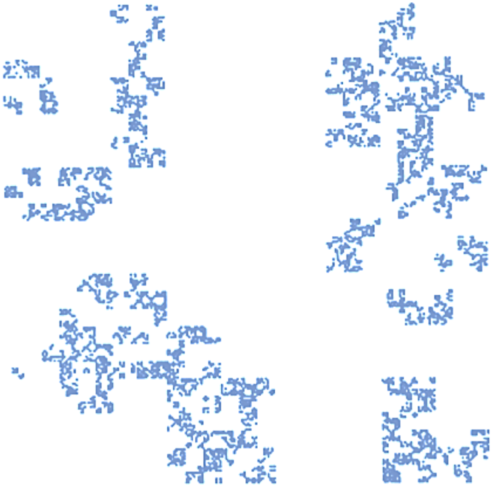

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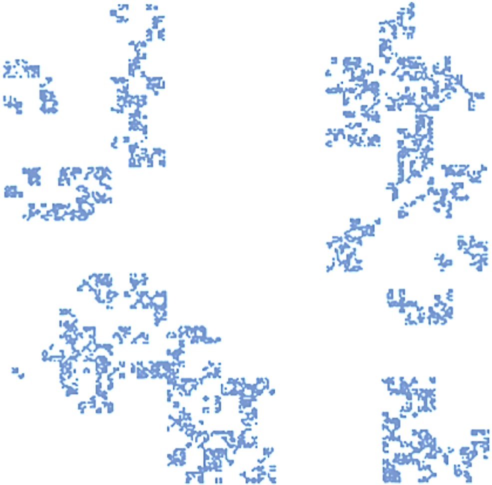


If $p > 1/M^2$ then $E_p \neq \emptyset$ with positive probability, conditional on which $\dim_H E_p = 2 + \log p / \log M$ almost surely.

Projections of Mandelbrot percolation

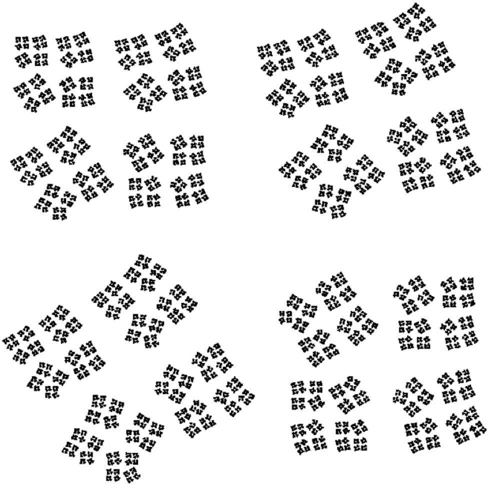
For Mandelbrot percolation assume $2 + \log p / \log M > 1$. Then conditional on $E_p \neq \emptyset$, almost surely:

• for all θ, $\text{proj}_\theta E_p$ contains an interval, so $\mathcal{L}(\text{proj}_\theta E_p) > 0$ (Rams & Simon, 2012)

• with μ the natural measure on E_p, for all θ, $\text{proj}_\theta \mu$ is absolutely continuous, with Hölder continuous density for all except the principal directions. (Peres & Rams, 2014)

• Mandelbrot percolation is a special case of a spatially independent martingale – A very general setting that covers projections of many sets and measures including variants on percolation, random cut-out sets and other random constructions. (Shmerkin & Soumala, 2015)

Percolation on self-similar sets



• We can run percolation on a self-similar set E. Assume that E has dense rotations.

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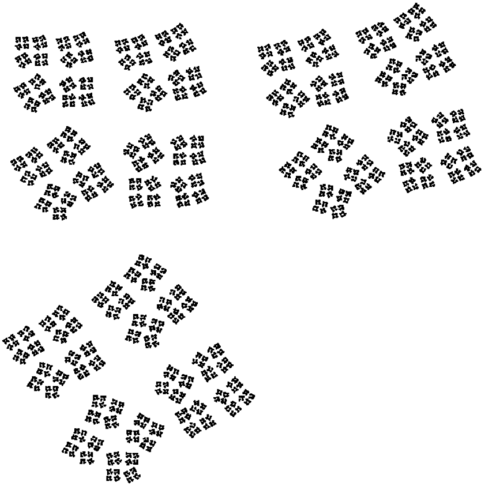

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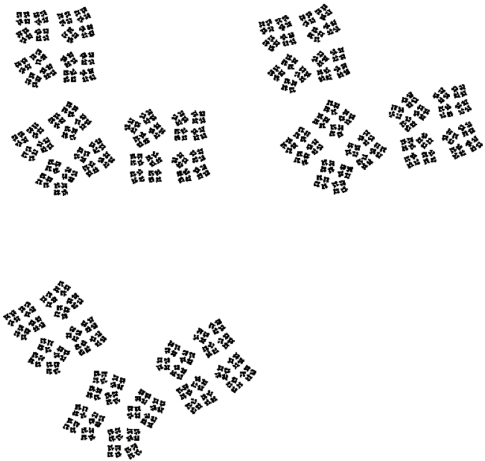

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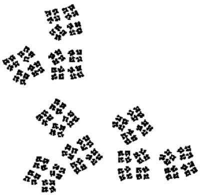

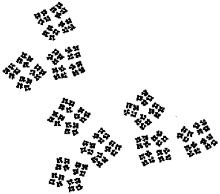

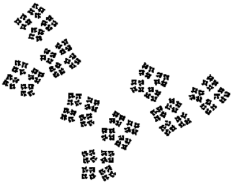

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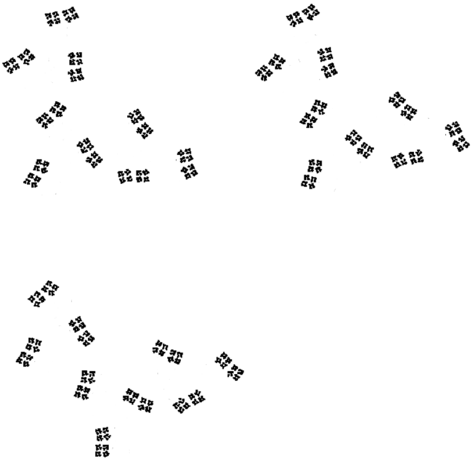

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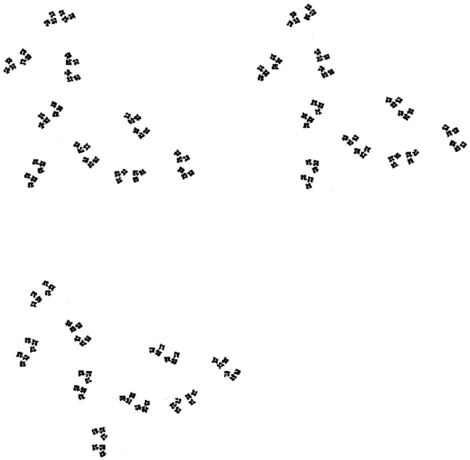

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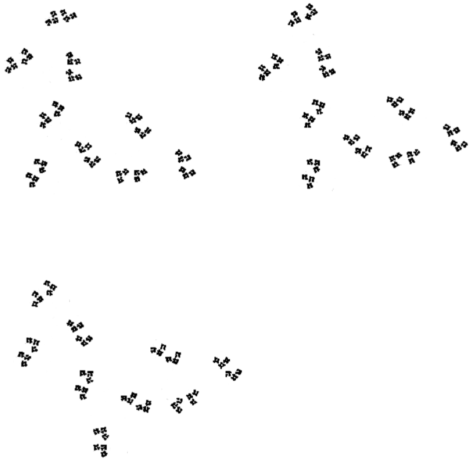


If $\dim_H E_p > 1$ then, almost surely, $\mathcal{L}(\mathrm{proj}_\theta E_p) > 0$ for all θ except for a set of θ of Hausdorff dimension 0. (F & Jin 2015)

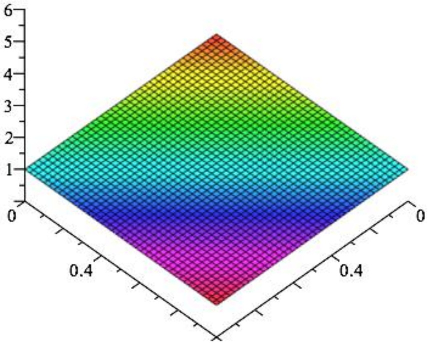
Random multiplicative cascades

• Random multiplicative cascades were introduced by Mandelbrot in 1974 in relation to fluid turbulence and studied by Kahane, Peyrière and others.

• Let W be a positive random variable with mean 1.

• Construct a sequence of random functions f_n on the unit square by repeatedly subdividing squares and multiplying the function on each subsquare by an independent realisation of W.

Multiplicative cascade construction on a square



• Squares are divided into 4 at each stage and the function on each subsquare multiplied by a independent realisation of W.

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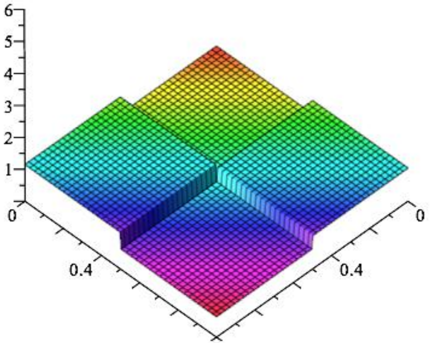

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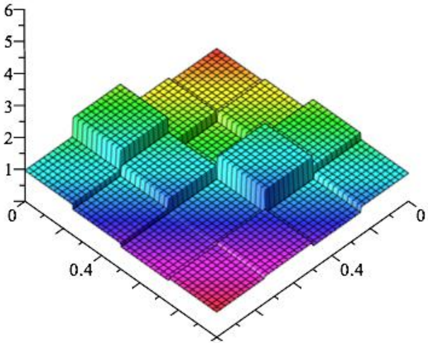

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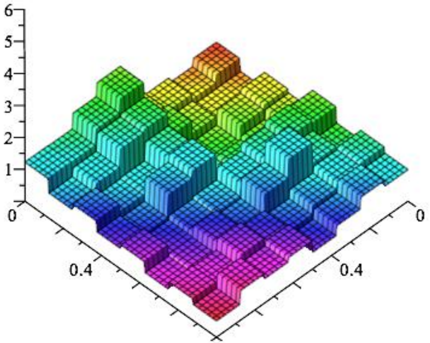

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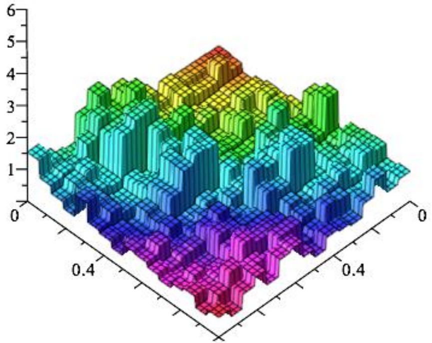

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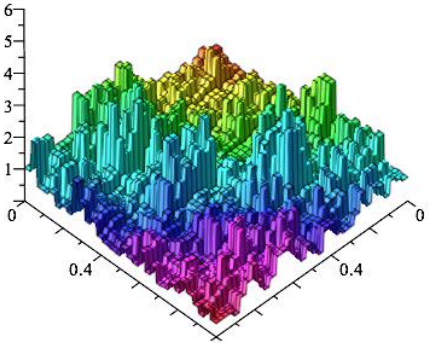

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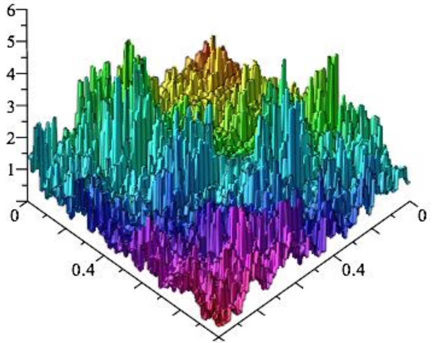

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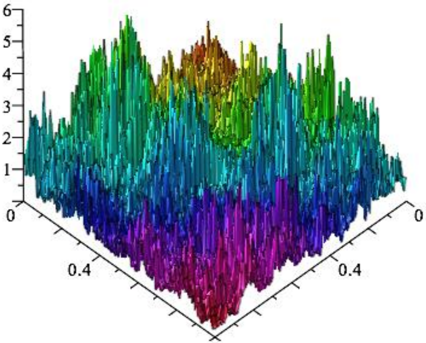

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Random multiplicative cascade on a square

• For each subset of the square A, the sequence $\mu_n(A) = \int_A f_n$ is a martingale, so with probability one, converges to $\mu(A)$. Then μ is a measure called a random multiplicative cascade measure.

Theorem (Shmerkin, Suomala, 2015) Let μ be a random cascade measure on the unit square. If $W \in (0, C]$ then almost surely $\mathrm{proj}_\theta \mu$ is absolutely continuous w.r.t Lebesgue measure for all $\theta \in [0, \pi)$.

Moreover, with the exception of the two prinicpal directions, the Radon-Nikodym derivatives are Hölder continuous.

A special case of spatially independent martingales.

Random multiplicative cascades

Properties of the random cascade measure μ:

• μ has a highly singular 'multifractal' structure.

• For a small region A
$\mu(A)$ is (approx) log-normally distributed, $\mathbb{E}(\mu(A)) =$ area(A).

• For small separated regions A, B correlations are very roughly

$$\mathrm{corr}(\log \mu(A), \log \mu(B)) \approx \mathrm{dist}(A, B)^{-\gamma}.$$

Drawbacks of μ:

• The construction involves preferred distance scales of 2^{-k}.

• Lack of spatial homogeneity - 'fault lines' between binary squares.

• Lack of isotropy - axis directions are special.

Is there a random mass distribution on a domain D with similar statistical characteristics but without these disadvantages, i.e. a construction that is 'continuous' rather than 'discrete'?

Overcoming the drawbacks

Around 1986 Kahane constructed such a process called Gaussian Multiplicative Chaos. His construction depended on divergent sums.

The construction was almost forgotten until around 2008 when Duplantier & Sheffield noticed it and termed the plane case Liouville quantum gravity measure on the domain D.

They also proposed an alternative construction of the same process using circle avarages of the Gaussian free field.

Gaussian Free Field

Let $D \subset \mathbb{R}^2$ be a 'nice' bounded domain. The Green function G_D on $D \times D$ is given by

$$G_D(x,y) = \log \frac{1}{|x-y|} - \mathbb{E}\bigg(\log \frac{1}{|E_D(x)-y|}\bigg),$$

The Green function is conformally invariant in the sense that if f is a conformal mapping, then

$$G_D(x,y) = G_{f(D)}(f(x), f(y)).$$

Let $\mathcal{M}$ be the vector space of signed measures on D such that $\int G_D(x,y)d|\mu|(x)d|\nu|(y) < \infty$. Then there exists mean zero real-valued Gaussian process $(\Gamma(\mu), \mu \in \mathcal{M})$ on $\mathcal{M}$ with covariance function

$$\mathbb{E}(\Gamma(\mu)\Gamma(\nu)) = \int_{D\times D} G_D(x,y)d\mu(x)d\nu(y).$$

Then Γ is the Gaussian Free Field on D.

Liouville quantum gravity

We would like to define a random measure $d\mu = e^{\gamma\Gamma(\delta_x)}dx$ which would have correlations 'like' the random cascade process. However, Γ is not a function but a distribution.

So we define a measure by approximation and taking a limit. For $x \in D$ and $\epsilon > 0$ let $\rho_{x,\epsilon}$ be normalized Lebesgue measure on $\{y \in D : |x - y| = \epsilon\}$, i.e., the circle centered at x with radius ϵ in D. Fix $\gamma \in [0, 2)$. For $\epsilon > 0$ define μ_ϵ by

$$\mu_\epsilon(A) = \epsilon^{\gamma^2/2} \int_A e^{\gamma\Gamma(\rho_{x,\epsilon})}\, dx. \tag{1}$$

Let $\mu = \text{weak-lim}_{\epsilon\to 0}\, \mu_\epsilon$.
Then μ exists and is non-degenerate almost surely and is called the γ-Liouville quantum gravity measure or γ-LQG measure on D.

Liouville quantum gravity - properties

For $\epsilon > 0$, $e^{\gamma\Gamma(\rho_{x,\epsilon})}$ has a lognormal distribution with

$$\mathbb{E}\big(e^{\gamma\Gamma(\rho_{x,\epsilon})}\big) = e^{\frac{\gamma^2}{2}\mathrm{Var}(\Gamma(\rho_{x,\epsilon}))} = \epsilon^{-\gamma^2/2}R(x,D)^{\gamma^2/2}$$ where

$R(x,D) \simeq \mathrm{dist}(x,\partial D)$ is the conformal radius of x in D. It follows that:

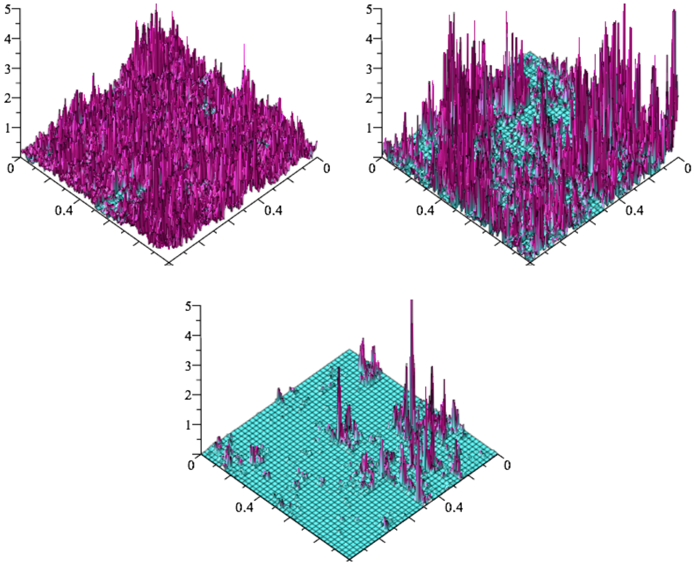
- $\mu(A)$ is close to log-normal if A is small with
$$\mathbb{E}(\mu(A)) \simeq \mathrm{dist}(A,\partial D)\ \mathrm{area}(A);$$

- For A, B small and separated,
$$\mathrm{corr}(\log\mu(A), \log\mu(B)) \approx \mathrm{dist}(A,B)^{-\gamma^2/2}.$$

- $\dim_H \mu = 2 - \frac{\gamma^2}{2}$.

- the construction of μ has no preferred scales, is (locally) spatially homogeneous, isotropic.

- the construction is conformally covariant.



Impressions of Liouville quantum gravity for $\gamma = 0.4, 1.4, 1.8$.

Why Liouville quantum gravity?

Mathematically:

• it leads to a very elegant theory - see N. Berestycki's notes.

• the conformal basis gives many nice properties with techniques from complex analysis available

• it is related to other random structures, such as limits of random graphs, circle packings, 'mating of Brownian trees', and SLE.

Physically:

• Quantum gravity models aim to give a space-time gravitational field valid at quantum scales when the field becomes highly distorted and distance only has meaning as a probability distribution.

• LQG defines a volume (area) measure in a 2-D model that has features that reflect what might be hoped for in a 4-D space-time model. The LQG measure may be regarded as representing the distortion of a smooth surface resulting from quantum effects.

Projections of Liouville quantum gravity

Theorem (F, Jin 2016) Let $0 < \gamma < \sqrt{2}$ and let μ be the LQG measure on a smooth domain D, so $\dim_H \mu = 2 - \gamma^2/2 > 1$. Then, almost surely, $\mathrm{proj}_\theta \mu$ is simultaneously absolutely continuous for all θ, with Radon-Nikodym derivative $f_\theta(x)$ satisfying a Hölder condition $|f_\theta(x) - f_\theta(y)| \leq |x - y|^\beta$ where

$$\beta = \left(\frac{1}{2\sqrt{2} + \sqrt{6 + 2\left(\sqrt{\frac{2}{\gamma^2}} - 1\right)^2}} \right)^2 \left(\sqrt{\frac{2}{\gamma^2}} - 1 \right)^2.$$

Corollary (Fourier transforms) For D a convex domain, almost surely, there is a (random) constant $C < \infty$ such that

$$|\widehat{\mu}(\xi)| \leq C|\xi|^{-\beta} \qquad (\xi \in \mathbb{R}^2),$$

i.e. $\dim_F \mu \geq 2\beta$.

Idea of proof Let ν_L be 1-D Lebesgue measure on the line L. Define random measures on lines using circle averages:

$$\widetilde{\nu}_{L,n}(dx) = 2^{-n\gamma^2/2} e^{\gamma \Gamma(\rho_{x,2^{-n}})} \nu_L(dx), \quad x \in L,$$

and let

$$Y_{L,n} := \widetilde{\nu}_{L,n}(L)$$

be the total mass of $\widetilde{\nu}_{L,n}$. We claim that a.s there is a C with

$$\sup_{n \geq 1} |Y_{L',n} - Y_{L,n}| \leq C \operatorname{dist}\big(L', L\big)^{\beta}. \quad (*)$$

Let $\widetilde{\nu}_L = \text{weak-lim}_{n\to\infty} \widetilde{\nu}_{L,n}$ be γ-LQG on ν_L and let $Y_L = \widetilde{\nu}_L(L)$ be its total mass.
Then

$$|Y_{L'} - Y_L| \leq C \operatorname{dist}\big(L', L\big)^{\beta}.$$

The conclusion follows since Y_L is the slice integral of μ.

The argument to obtain

$$\sup_{n\geq 1}|Y_{L',n} - Y_{L,n}| \leq C \text{ dist}\big(L', L\big)^{\beta} \quad (*)$$

is reminiscent of that of the Kolmogorov-Chentsov continuity theorem. It combines two estimates: given $p, q > 1$.

$$\mathbb{E}\big(|Y_{L,n+1} - Y_{L,n}|^{p}\big) \leq C\ 2^{-\alpha n}$$

and

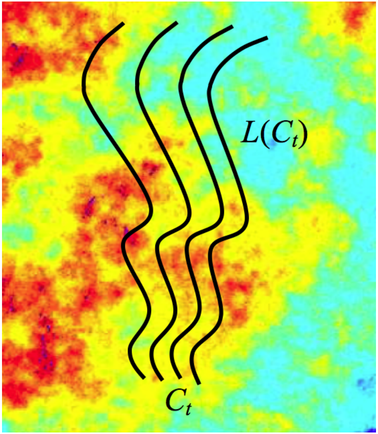
$$\mathbb{E}\big(\max_{1\leq k\leq n}|Y_{L',k} - Y_{L,k}|^{q}\big) \leq C \text{ dist}\big(L', L\big)^{\lambda} 2^{\alpha' n}.$$

where $\alpha, \alpha', \lambda > 0$ depend on p and q. By choosing suitable values of the exponents we get $(*)$.

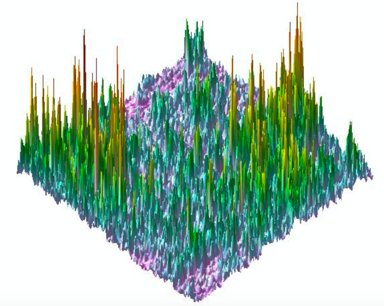
Properties of the LQG measure

More generally, if $0 < \gamma < \sqrt{2}$ we may define the (random) quantum length $L_q(C)$ of a curve C by letting ν be length measure on C, letting
$\widetilde{\nu}_\epsilon(dl) = \epsilon^{\gamma^2/2} e^{\gamma \Gamma(\rho_{x,\epsilon})}\, \nu(dl)$,
$\widetilde{\nu} = \text{weak-lim}_{\epsilon \to 0}\, \widetilde{\nu}_\epsilon$ and
$L_q(C) = \widetilde{\nu}(D)$.

Theorem (F, Jin, 2016) If $0 < \gamma < \sqrt{2}$, then given any (reasonable) parameterised family of curves C_t, with probability 1 the quantum length $L_q(C_t)$ is defined for all t and varies (Hölder) continuously with t.



Liouville quantum gravity is currently of great interest, not least because of its many relationships to other areas of maths and probability.



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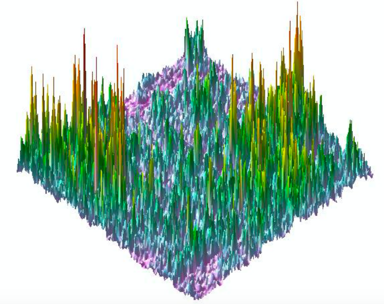

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